Reducible Differential Equations

Questions

Q1.

The concentration of a drug in the bloodstream of a patient, *t* hours after the drug has been administered, where t ≤ 6, is modelled by the differential equation

$$t^2\frac{d^2C}{dt^2} - 5t\frac{dC}{dt} + 8C = t^3 \qquad \text{(I)}$$

where *C* is measured in micrograms per litre.

(a) Show that the transformation $t = e^x$ transforms equation (I) into the equation

$$\frac{d^2C}{dx^2} - 6\frac{dC}{dx} + 8C = e^{3x} \qquad \text{(II)}$$

(5)

(b) Hence find the general solution for the concentration *C* at time *t* hours.

(7)

Given that when $t = 6$, $C = 0$ and $\frac{dC}{dt} = -36$

(c) find the maximum concentration of the drug in the bloodstream of the patient.

(5)

(Total for question = 17 marks)

Q2.

A vibrating spring, fixed at one end, has an external force acting on it such that the centre of the spring moves in a straight line. At time *t* seconds, $t \geq 0$, the displacement of the centre *C* of the spring from a fixed point *O* is *x* micrometres.

The displacement of *C* from *O* is modelled by the differential equation

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + (2 + t^2)x = t^4 \qquad \text{(I)}$$

(a) Show that the transformation $x = tv$ transforms equation (I) into the equation

$$\frac{d^2v}{dt^2} + v = t \qquad \text{(II)}$$

(5)

(b) Hence find the general equation for the displacement of *C* from *O* at time *t* seconds.

(7)

(c) (i) State what happens to the displacement of *C* from *O* as *t* becomes large.

(ii) Comment on the model with reference to this long term behaviour.

(2)

(Total for question = 14 marks)

Q3.

(a) Show that the substitution $x = e^u$ transforms the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = -x^{-2}, \quad x > 0 \qquad \text{(I)}$$

into the equation

$$\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = -e^{-2u} \qquad \text{(II)}$$

(6)

(b) Find the general solution of the differential equation (II).

(7)

(c) Hence obtain the general solution of the differential equation (I) giving your answer in the form $y = f(x)$

(1)

(Total for question = 14 marks)

Mark Scheme – Reducible Differential Equations

Q1.

Question	Scheme	Marks	AOs
(a)	**Examples:** $t = e^x \Rightarrow \frac{dt}{dC} = e^x \frac{dx}{dC}$ or $\frac{dC}{dx} = t\frac{dC}{dt}$ or $\frac{dC}{dt} = e^{-x}\frac{dC}{dx}$ or $\frac{dC}{dt} = \frac{1}{t}\frac{dC}{dx}$	M1	1.1b
	E.g. $\frac{dC}{dx} = t\frac{dC}{dt} \Rightarrow \frac{d^2C}{dx^2} \times \frac{dx}{dt} = t\frac{d^2C}{dt^2} + \frac{dC}{dt}$	**d**M1 A1	2.1 1.1b
	$\frac{d^2C}{dx^2} \times \frac{1}{t} = t\frac{d^2C}{dt^2} + \frac{1}{t}\frac{dC}{dx} \Rightarrow t^2\frac{d^2C}{dt^2} = \frac{d^2C}{dx^2} - \frac{dC}{dx}$ $t^2\frac{d^2C}{dt^2} - 5t\frac{dC}{dt} + 8C = \frac{d^2C}{dx^2} - \frac{dC}{dx} - 5\frac{dC}{dx} + 8C$	**d**M1	2.1
	$\frac{d^2C}{dx^2} - 6\frac{dC}{dx} + 8C = e^{3x}$ *	A1*	1.1b
		(5)	
	Mark (b) and (c) together and ignore labelling		
(b)	$m^2 - 6m + 8 = 0 \Rightarrow m = 2, 4$	M1	1.1b
	$(C =) Ae^{4x} + Be^{2x}$	A1ft	1.1b
	PI is $C = ke^{3x}$	B1	2.2a
	$\frac{dC}{dx} = 3ke^{3x}, \frac{d^2C}{dx^2} = 9ke^{3x} \Rightarrow 9k - 18k + 8k = 1 \Rightarrow k = -1$	M1	1.1b
	$C = Ae^{4x} + Be^{2x} - e^{3x}$	A1	1.1b
	$t = e^x \Rightarrow C = \ldots$	M1	3.4
	$C = At^4 + Bt^2 - t^3$	A1	2.2a
		(7)	
(c)	$t = 6, C = 0 \Rightarrow 1296A + 36B - 216 = 0$	M1	3.4
	$\frac{dC}{dt} = 4At^3 + 2Bt - 3t^2 \Rightarrow -36 = 864A + 12B - 108$	M1	3.4
	$A = 0, B = 6 \Rightarrow C = 6t^2 - t^3$	A1	1.1b
	$\frac{dC}{dt} = 12t - 3t^2 = 0 \Rightarrow t = 4 \Rightarrow C = \ldots$	**dd**M1	1.1b
	$C = 6(4)^2 - (4)^3 = 32\mu gL^{-1}$	A1	3.2a
		(5)	
			(17 marks)

Notes

(a)

M1: Uses $t = e^x$ to obtain a correct equation in terms of $\frac{dC}{dx}$, $\frac{dC}{dt}$ and $t\left(\text{or } e^x\right)$ or their reciprocals

dM1: Differentiates again **correctly** with the product rule and chain rule in order to obtain an equation involving $\frac{d^2C}{dt^2}$ and $\frac{d^2C}{dx^2}$. **This needs to be fully correct calculus work allowing sign errors only.**

A1: Correct equation.

dM1: Shows clearly their substitution into the differential equation (or equivalent work) in order to form the new equation. **Dependent on the first method mark and dependent on having obtained two terms for the second derivative.**

Allow substitution for $\frac{dC}{dx}$ and $\frac{d^2C}{dx^2}$ into equation (II) to achieve equation (I)

A1*: Fully correct proof with no errors

(b)

M1: Forms and solves a quadratic auxiliary equation $m^2 - 6m + 8 = 0$

A1ft: Correct form for the CF for their AE solutions **which must be distinct and real**

B1: Deduces the correct form for the PI (ke^{3x})

M1: Differentiates their PI, **which is of the correct form**, and substitutes their derivatives into the DE to find "k"

A1: Correct GS for C in terms of x (**this must be seen explicitly unless implied by subsequent work**)

M1: Links the solution to DE (II) to the solution of the model to find the concentration at time t

A1: Deduces the correct GS for the concentration

If a correct GS is fortuitously found in (b) (e.g. from an incorrect PI form, allow full recovery in (c).

(c)

M1: Uses the conditions of the model ($t = 6$, $C = 0$) to form an equation in A and B.

***Note that <u>is</u> acceptable to use their C in terms of x for this mark as long as they use $x = \ln 6$ when $C = 0$

M1: Uses the conditions of the model $\left(t = 6, \frac{dC}{dt} = -36\right)$ to form another equation in A and B.

***Note that it is <u>**not**</u> acceptable to use $\frac{dC}{dx} = -36$ with $x = \ln 6$, as it is necessary to use $\frac{dC}{dt} = \frac{dC}{dx}\frac{dx}{dt}$ e.g. $-36 = \left(4Ae^{4\ln 6} + 2Be^{2\ln 6} - 3e^{3\ln 6}\right) \times e^{-\ln 6}$ or $-216 = 4Ae^{4\ln 6} + 2Be^{2\ln 6} - 3e^{3\ln 6}$

A1: Correct equation connecting C with t

ddM1: Uses a suitable method to find the maximum concentration. E.g. solves $\frac{dC}{dt} = 0$ for t and substitutes to find C. Allow a solution that solves $\frac{dC}{dx} = 0$ for x and uses this correctly to find C.

Dependent on <u>both</u> previous method marks.

A1: Obtains 32 μgL^{-1} using the model. Units are required but allow e.g.

- micrograms per litre
- μg/L
- μg/l
- μgl^{-1}

Q2.

Question	Scheme	Marks	AOs
(a)	Use of $x = tv$ to give $\frac{dx}{dt} = v + t\frac{dv}{dt}$	M1	1.1b
	Hence $\frac{d^2x}{dt^2} = \frac{dv}{dt} + \frac{dv}{dt} + t\frac{d^2v}{dt^2}$	M1	2.1
		A1	1.1b
	Uses $t^2\left(\text{their 2}^{\text{nd}}\text{derivative}\right) - 2t\left(\text{their 1}^{\text{st}}\text{derivative}\right) + (2+t^2)x = t^4$ and simplifies LHS.	M1	2.1
	$\left(t^3\frac{d^2v}{dt^2} + t^3v = t^4 \text{ leading to}\right)$ $\frac{d^2v}{dt^2} + v = t$ *	A1*	1.1b
		(5)	
(b)	Solve $\lambda^2 + 1 = 0$ to give $\lambda^2 = -1$	M1	1.1b
	$v = A\cos t + B\sin t$	A1ft	1.1b
	Particular Integral is $v = kt + l$	B1	2.2a
	$\frac{dv}{dt} = k$ and $\frac{d^2v}{dt^2} = 0$ and solve $0 + kt + l = t$ to give $k = 1$, $l = 0$	M1	1.1b
	Solution: $v = A\cos t + B\sin t + t$	A1	1.1b
	Displacement of C from O is given by $x = tv = ...$	M1	3.4
	$x = t(A\cos t + B\sin t + t)$	A1	2.2a
		(7)	
(c)(i)	For large t, the displacement gets very large (and positive).	B1	3.2a
(ii)	Model suggests midpoint of spring moving relative to fixed point has large displacement when t is large, which is unrealistic. The spring may reach elastic limit / will break.	B1	3.5a
		(2)	
		(14 marks)	

Notes	
(a)	
M1	Uses product rule to obtain first derivative.
M1	Continues to differentiate again, with product rule and chain rule as appropriate, in order to establish the second derivative.
A1	Correct second derivative. Accept equivalent expressions.
M1	Shows clearly the substitution into the given equation in order to form the new equation and gathers like terms.
A1*	Fully correct solution leading to the given answer.
(b)	Accept variations on symbols for constants throughout.
M1	Form and solve a quadratic Auxiliary Equation.
A1ft	Correct form of the Complementary Function for their solutions to the AE.
B1	Deduces the correct form for the Particular Integral (note $v = mt^2 + kt + l$ is fine).
M1	Differentiates their Particular Integral and substitutes their derivatives into the equation to find the constants ($m = 0$ if used).
A1	Correct general solution for equation (II).
M1	Links the solution to equation (II) to the solution of the model equation correctly to find the displacement equation.
A1	Deduces the correct general solution for the displacement.

Notes continued	
(c)(i)	
B1	States that for large t the displacement is large o.e. Accept e.g. as $t\to\infty$, $x\to\infty$.
(c)(ii)	
B1	Reflect on the context of the original problem. Accept 'model unrealistic' / 'spring will break'.

Q3.

Question Number	Scheme	Marks
(a)	$x=\mathrm{e}^u \quad \frac{\mathrm{d}x}{\mathrm{d}u}=\mathrm{e}^u$ or $\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^{-u}$ or $\frac{\mathrm{d}x}{\mathrm{d}u}=x$ or $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$	
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^{-u}\frac{\mathrm{d}y}{\mathrm{d}u}$	M1A1
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{e}^{-u}\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}u}+\mathrm{e}^{-u}\frac{\mathrm{d}^2y}{\mathrm{d}u^2}\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^{-2u}\left(-\frac{\mathrm{d}y}{\mathrm{d}u}+\frac{\mathrm{d}^2y}{\mathrm{d}u^2}\right)$	M1A1
	$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y=-x^{-2}$	
	$\mathrm{e}^{2u}\times\mathrm{e}^{-2u}\left(-\frac{\mathrm{d}y}{\mathrm{d}u}+\frac{\mathrm{d}^2y}{\mathrm{d}u^2}\right)-2\mathrm{e}^u\times\mathrm{e}^{-u}\frac{\mathrm{d}y}{\mathrm{d}u}+2y=-\mathrm{e}^{-2u}$	dM1
	$\frac{\mathrm{d}^2y}{\mathrm{d}u^2}-3\frac{\mathrm{d}y}{\mathrm{d}u}+2y=-\mathrm{e}^{-2u}$ *	A1cso (6)
(a)		
M1	obtaining $\frac{\mathrm{d}y}{\mathrm{d}x}$ using chain rule here or seen later (may not be shown explicitly but appear in the substitution)	
A1	correct expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ any equivalent form (again, may not be seen until substitution)	
M1	obtaining $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ using product rule (penalise lack of chain rule by the A mark)	
A1	a correct expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ any equivalent form	
dM1	substituting in the equation to eliminate x **Only** u and y now Depends on both previous M marks. Substitution must have come from their work	
A1cso	obtaining the given result from completely correct work.	

	ALTERNATIVE 1	
	$x = e^u \quad \frac{dx}{du} = e^u = x$	
	$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = x\frac{dy}{dx}$	M1A1
	$\frac{d^2y}{du^2} = 1\frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$	M1A1
	$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$	
	$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) - 2x \times \frac{1}{x}\frac{dy}{du} + 2y = -x^{-2}$	
	$\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = -e^{-2u}$ *	dM1A1cso (6)
M1	obtaining $\frac{dy}{du}$ using chain rule here or seen later	
A1	correct expression for $\frac{dy}{du}$ here or seen later	
M1	obtaining $\frac{d^2y}{du^2}$ using product rule (penalise lack of chain rule by the A mark)	
A1	Correct expression for $\frac{d^2y}{du^2}$ any equivalent form	
dM1A1cso	As main scheme	

	ALTERNATIVE 2:	
	$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$	
	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x}\frac{dy}{du}$	M1A1
	$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{d^2y}{du^2} \times \frac{du}{dx} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2y}{du^2}$	M1A1
	$x^2\left(-\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2y}{du^2}\right) - 2x \times \frac{1}{x}\frac{dy}{du} + 2y = -x^{-2}$	
	$\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = -e^{-2u}$ * Depends on both previous M marks	dM1A1cso

	There are also **other solutions** which will appear, either starting from equation II and obtaining equation I, or mixing letters x, y and u until the final stage.	
M1	obtaining a first derivative with chain rule	
A1	correct first derivative	
M1	obtaining a second derivative with product rule (Chain rule errors are penalised through A marks)	
A1	correct second derivative with 2 or 3 variables present	
dM1	Either substitute in equation I or substitute in equation II according to method chosen **AND** obtain an equation with only y and u (following sub in eqn I) or with only x and y (following sub in eqn II)	
A1cso	Obtaining the required result from completely correct work	

Question Number	Scheme	Notes	Marks
(b)	$m^2-3m+2=0 \Rightarrow m=1,2$	M1: Forms AE and attempts to solve to $m=$ ...or values seen in CF A1: Both values correct. May only be seen in the CF	M1A1
	$(\text{CF}=)\, Ae^u+Be^{2u}$	CF correct oe can use any (single) variable	A1
	$y=\lambda e^{-2u}$		
	$\frac{dy}{du}=-2\lambda e^{-2u}$ $\frac{d^2y}{du^2}=4\lambda e^{-2u}$	PI of form $y=\lambda e^{-2u}$ (or $y=\lambda u e^{-2u}$ if $m=-2$ is a solution of the aux equation) **and** differentiate PI twice wrt u. Allow with x instead of u	M1
	$4\lambda e^{-2u}+6\lambda e^{-2u}+2\lambda e^{-2u}=-e^{-2u}$ $\Rightarrow \lambda=-\frac{1}{12}$	dM1 substitute in the equation to obtain value for λ Dependent on the second M1 A1 $\lambda=-\frac{1}{12}$	dM1A1
	$y=Ae^u+Be^{2u}-\frac{1}{12}e^{-2u}$	A complete solution, follow through their CF and PI. Must have $y=$ a function of u Allow recovery of incorrect variables.	B1ft
			(7)
(c)	$y=Ax+Bx^2-\frac{1}{12x^2}$ Or $y=Ae^{\ln x}+Be^{2\ln x}-\frac{1}{12e^{2\ln x}}$	Reverse the substitution to obtain a correct expression for y in terms of x No ft here $\frac{1}{12x^2}$ or $\frac{1}{12}x^{-2}$ Must start $y-$ ...	B1
			(1)
			Total 14